MA 441, Fall 2014

Assignment $\frac{9}{2}$

This homework is due *Thursday* Oct 16.

This homework is worth one half of normal homework in terms of course grade, and is not included in denominator of your course grade. That is, it's a freebie.

You are given exercises from Kropholler's book *Analysis from Scratch.* In each item in each exercise 2.6–2.12, give an answer (True/False). You can use question sheets and just circle/underline your answers. In *first seven* items of 2.12 supply proofs.

You don't need to do exercise 2.13 (more exactly, as mentioned above, you only have to do it for first seven items of 2.12) but you are welcome to read the explanation after this exercise.

Each answer is worth 1 point, each proof is worth 1 point (making 2 points for the items with proofs). Together there are 83 items, making a total of 90 points, if I didn't miscount. 83 points is considered 100%. If you go above 83, you will get over 100% for this assignment and it will count towards your course grade.

The remaining exercises in this section are five finger exercises to test understanding of definitions and notation. "Five finger" does not mean easy: some are easy, some less so. First some exercises involving one quantifier:

EXERCISE 2.6. Inequalities quantified by $\forall$ or $\exists$. Determine whether they are true or false.

True / False $\quad \forall\, x \in \mathbb{R},\ x^2 > 0.$

True / False $\quad \forall\, y \in \mathbb{R},\ y^3 \leq 0.$

True / False $\quad \forall\, x \in \mathbb{R},\ x^3 \geq 0.$

True / False $\quad \forall\, x \in \mathbb{R},\ x^2 \geq 0.$

True / False $\quad \forall\, x, y \in \mathbb{R},\ x^2 + y^2 > 0.$

True / False $\quad \forall\, y, z \in \mathbb{R},\ y^2 + z^4 \geq 0.$

True / False $\quad \forall\, x \in \mathbb{R},\ x^2 - 2x + 1 > 0.$

True / False $\quad \forall\, x \in \mathbb{R},\ x^2 < 0.$

True / False $\quad \forall\, x, y \in \mathbb{R},\ x^2 + y^2 < 0.$

True / False $\quad \forall\, y \in \mathbb{R},\ 2y^2 - 3y + 1 \geq 0.$

True / False $\quad \exists\, x \in \mathbb{R},\ x^2 \leq 0.$

True / False $\quad \exists\, y \in \mathbb{R},\ y^3 \leq 0.$

True / False $\quad \exists\, x \in \mathbb{R},\ x^2 > 0.$

True / False $\quad \exists\, x, y \in \mathbb{R},\ x^2 + y^3 < 0.$

True / False $\quad \exists\, x \in \mathbb{R},\ x^5 < 0.$

True / False $\quad \exists\, x \in \mathbb{R},\ x^2 + x + 1 < 0.$

True / False $\quad \exists\, x \in \mathbb{R},\ 2x^2 - 3x + 1 < 0.$

True / False $\quad \exists\, y \in \mathbb{R},\ y^2 + y + 1 > 0.$

True / False $\quad \exists\, y \in \mathbb{R},\ y^2 - 2y + 1 < 0.$

True / False $\quad \exists\, y \in \mathbb{R},\ 2y^2 - 3y + 1 \geq 0.$

EXERCISE 2.7. More inequalities quantified by $\forall$ or $\exists$. Determine whether they are true or false.

True / False $\quad \forall\, x \in \mathbb{R},\ x^2 \geq 0.$

True / False $\quad \forall\, y \in \mathbb{R},\ y^3 \geq 0.$

True / False $\quad \forall\, x \in \mathbb{R},\ x^3 \leq 0.$

True / False $\quad \forall\, x \in \mathbb{R},\ x^2 > -1.$

True / False $\quad \forall\, x,y \in \mathbb{R},\ x^2+y^2 \geq 0.$

True / False $\quad \forall\, y,z \in \mathbb{R},\ y^2+z^4 > 0.$

True / False $\quad \forall\, x \in \mathbb{R},\ x^2-2x+1 \geq 0.$

True / False $\quad \forall\, x \in \mathbb{R},\ x^2 \leq 0.$

True / False $\quad \forall\, x,y \in \mathbb{R},\ x^2+y^2-2 > 1.$

True / False $\quad \forall\, y \in \mathbb{R},\ 2y^2-3y+1 > 0.$

True / False $\quad \exists\, x \in \mathbb{R},\ x^2 \geq 0.$

True / False $\quad \exists\, y \in \mathbb{R},\ y^3 < 0.$

True / False $\quad \exists\, x \in \mathbb{R},\ x^2 < 0.$

True / False $\quad \exists\, x,y \in \mathbb{R},\ x^2+y^3 \leq 0.$

True / False $\quad \exists\, x \in \mathbb{R},\ x^3 < 0.$

True / False $\quad \exists\, x \in \mathbb{R},\ x^2-x+1 < 0.$

True / False $\quad \exists\, x \in \mathbb{R},\ 2x^2+3x+1 < 0.$

True / False $\quad \exists\, y \in \mathbb{R},\ y^2-y+1 > 0.$

True / False $\quad \exists\, y \in \mathbb{R},\ y^2-2y+1 \leq 0.$

True / False $\quad \exists\, y \in \mathbb{R},\ 2y^2-3y+1 \leq 0.$

EXERCISE 2.8. $\forall$ quantified statements concerning logical implication.

True / False $\forall\, x \in \mathbb{R},\ (x > 0 \implies x^2 > 0).$

True / False $\forall\, y \in \mathbb{R},\ (y^3 < 0 \implies y < 0).$

True / False $\forall\, x \in \mathbb{R},\ (x^2 > 0 \implies x > 0).$

True / False $\forall\, x \in \mathbb{R},\ (x^3 > 0 \implies x > 0).$

True / False $\forall\, x, y \in \mathbb{R},\ (x < y \implies x^2 < y^2).$

True / False $\forall\, y, z \in \mathbb{R},\ (y^2 < z^2 \implies y < z).$

True / False $\forall\, x \in \mathbb{R},\ (x^2 - 2x + 1 > 0 \implies x \neq 0).$

True / False $\forall\, x \in \mathbb{R},\ (x^2 < 4 \implies x < 2).$

True / False $\forall\, y \in \mathbb{R},\ (2y^2 - 3y + 1 > 0 \implies \frac{1}{2} < y < 1).$

True / False $\forall\, w, x, y, z \in \mathbb{R},\ (wz^2 + xz + y = 0 \implies x^2 - 4wy \geq 0).$

EXERCISE 2.9.

True / False $\exists a, b, c \in \mathbb{N}$ s.t. $a^2 + b^2 = c^2$.

True / False $\exists a, b, c \in \mathbb{N}$ s.t. $a^3 + b^3 = c^3$.

True / False $\forall n \in \mathbb{N},\ n^2 - n + 41$ is a prime number.

Next we come to exercises involving two quantifiers. Imagine a two-player game: you play the $\exists$ moves and your opponent plays the $\forall$ moves. I like to imagine a game between the Devil ($\forall$) and myself ($\exists$), others prefer the analogy with Abelard and Hélouise (i.e. $\forall$belard et H$\exists$louise). Some are satisfied with no analogy.

EXERCISE 2.10. $\forall\exists$ and $\exists\forall$ quantified statements. Which are true?

True / False $\forall\, x \in \mathbb{R}\quad \exists\, y \in \mathbb{R}$ such that $x > y$.

True / False $\forall\, y \in \mathbb{R}\quad \exists\, x \in \mathbb{R}$ such that $x > y$.

True / False $\forall\, x \in \mathbb{R}\quad \exists\, y \in \mathbb{R}$ such that $y - x^2 > 1000$.

True / False $\forall\, y \in \mathbb{R}\quad \exists\, x \in \mathbb{R}$ such that $x^3 = y^2$.

True / False $\forall\, x \in \mathbb{R}\quad \exists\, y \in \mathbb{R}$ such that $x^3 = y^2$.

True / False $\exists\, x \in \mathbb{R}$ such that $\forall\, y \in \mathbb{R}\ \ y < x$.

True / False $\exists\, y \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R}\ \ x^2 + yx - 2 > 0$.

True / False $\exists\, m \in \mathbb{N}$ such that $\forall\, n \geq m\quad \dfrac{1}{3} - \dfrac{n^2 + 7}{3n^2 + n + 1} < 10^{-9}$.

True / False $\exists\, m \in \mathbb{N}$ such that $\forall\, n \geq m\quad \dfrac{1}{3} - \dfrac{9n^2 + 7}{3n^2 + n + 1} < 10^{-9}$.

True / False $\exists\, m \in \mathbb{N}$ such that $\forall\, n \geq m\quad \sqrt{n+1} - \sqrt{n} < 10^{-12}$.

EXERCISE 2.11. More $\forall\exists$ and $\exists\forall$ quantified statements. Which are true?

True / False $\quad \forall\, x \in \mathbb{R} \quad \exists\, y \in \mathbb{R}$ such that $x < y$.

True / False $\quad \forall\, y \in \mathbb{R} \quad \exists\, x \in \mathbb{R}$ such that $x < y$.

True / False $\quad \forall\, x \in \mathbb{R} \quad \exists\, y \in \mathbb{R}$ such that $y - x^2 < 1000$.

True / False $\quad \forall\, y \in \mathbb{R} \quad \exists\, x \in \mathbb{R}$ such that $x^2 < y^3$.

True / False $\quad \forall\, x \in \mathbb{R} \quad \exists\, y \in \mathbb{R}$ such that $x^3 < y^2$.

True / False $\quad \exists\, x \in \mathbb{R}$ such that $\quad \forall\, y \in \mathbb{R} \;\; y \leq x$.

True / False $\quad \exists\, x \in \mathbb{R}$ such that $\quad \forall\, y \in \mathbb{R} \;\; x^2 + yx - 2 \geq 0$.

True / False $\quad \exists\, m \in \mathbb{N}$ such that $\quad \forall\, n \geq m \quad \left|\frac{1}{3} - \frac{n^2+7}{3n^2+n+1}\right| < 10^{-9}$.

True / False $\quad \exists\, m \in \mathbb{N}$ such that $\quad \forall\, n \geq m \quad \left|\frac{1}{3} - \frac{9n^2+7}{3n^2+n+1}\right| < 10^{-9}$.

True / False $\quad \exists\, m \in \mathbb{N}$ such that $\quad \forall\, n \geq m \quad \sqrt{n+1} - \sqrt{n} < 10^{-20}$.

EXERCISE 2.12. Yet more $\forall\exists$ and $\exists\forall$ quantified statements. Which are true?

True / False $\quad \forall\, x \in \mathbb{R} \quad \exists\, y \in \mathbb{R}$ such that $x^2 < y$.

True / False $\quad \forall\, y \in \mathbb{R} \quad \exists\, x \in \mathbb{R}$ such that $x^2 < y$.

True / False $\quad \forall\, x \in \mathbb{R} \quad \exists\, y \in \mathbb{R}$ such that $y^2 + x < 1000$.

True / False $\quad \forall\, y \in \mathbb{R} \quad \exists\, x \in \mathbb{R}$ such that $xy > 0$.

True / False $\quad \forall\, x \in \mathbb{R} \quad \exists\, y \in \mathbb{R}$ such that $xy \geq 0$.

True / False $\quad \exists\, x \in \mathbb{R}$ such that $\quad \forall\, y \in \mathbb{R} \;\; y^2 \leq x$.

True / False $\quad \exists\, x \in \mathbb{R}$ such that $\quad \forall\, y \in \mathbb{R} \;\; x^2 + y^2x + 2 \geq 0$.

True / False $\quad \exists\, m \in \mathbb{N}$ such that $\quad \forall\, n \geq m \quad \left|3 - \frac{n^2+7}{3n^2+n+1}\right| < 10^{-9}$.

True / False $\quad \exists\, m \in \mathbb{N}$ such that $\quad \forall\, n \geq m \quad \left|3 - \frac{9n^2+7}{3n^2+n+1}\right| < 10^{-9}$.

True / False $\quad \exists\, m \in \mathbb{N}$ such that $\quad \forall\, n \geq m \quad \sqrt{n^2+n} - \sqrt{n^2} < 10^{-20}$.